Complete Markets - a Wishart based approach

Romano Trabalzini

joint work with
Prof. Mark H. A. Davis
Department of Mathematics
Imperial College of Science and Technology

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MARKET COMPLETION & SPANNING SET OF OPTIONS

- References:
  - [Ross, 1976]
  - [Romano and Touzi, 1997]
  - [Davis, 2004]
  - [Davis and Obłój, 2008]

- It is well known that the Black-Scholes financial market model, consisting of a log-normal asset-price diffusion and a non-random money market account, is complete: every contingent claim is replicated by a portfolio formed by dynamic trading in the two assets.
- Ultimately this result rests on the martingale representation property of Brownian motion.
- Upon attempting to correct the empirical deficiencies of the asset model by including, say, stochastic volatility, completeness is lost if the original two assets are regarded as the only tradables: there are no longer enough assets to span the market.
- However there are traded options markets for many assets such as single stocks or stock indices, so it is a natural question to ask whether the market becomes complete when these are included.
- An early result in this direction was provided by [Romano and Touzi, 1997] who showed that a single call option completes the market when there is stochastic volatility driven by one extra Brownian motion.
- Providing a general theory has been very problematic.

- There are two main approaches, labeled martingale models and market models by [Schweizer and Wissel, 2008b].
- In a market model ([Schönbucher, 1999] , [Schweizer and Wissel, 2008a], [Schweizer and Wissel, 2008b]) one specifies directly the price processes of all traded assets, be they underlying assets or derivatives. For the latter, say a call option with strike K and exercise time T on an asset S_t, it is generally more convenient to model a proxy such as the implied volatility $\hat{\sigma}_t$ which is related in a one-to-one way to the price process A_t of the call by $A_t = \mathrm{BS}(S_t, K, r, \hat{\sigma}_t, T - t)$, where $BS(\cdot,\cdot,\cdot,\cdot,\cdot)$ is the Black-Scholes formula.
- In the martingale model ([Davis, 2004],[Davis and Obłój, 2008] and the present work) one starts with a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{Q})$: $\mathbb{Q}$ is a risk-neutral measure, so all discounted asset prices are $\mathbb{Q}$-martingales which can be constructed by conditional expectation: the price process for an asset that has the integrable $\mathcal{F}_T$-measurable value H at some final time T is $S_t = \mathbf{E}[e^{-r(T-t)}H|\mathcal{F}_t]$ for $t < T$, where r is the riskless interest rate. The distinction between an 'underlying asset' and a 'contingent claim' largely disappears in this approach. A specific model is obtained by specifying some process whose natural filtration is $(\mathcal{F}_t)$, for example a diffusion process.

- work on probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{Q})$ - usual conditions; let $D \subset \mathbb{R}^n$
- the martingale measure $\mathbb{Q}$ is the only one taken in consideration in this work
- consider the following vector valued SDE - "the factor process":

$$d\xi_t = \mu(t,\xi_t)dt + \sigma(t,\xi_t)dW_t, \qquad \xi_0 = \tilde{\xi}_0 \in D \tag{1}$$

where $\mu : \mathbb{R}_+ \times D \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times D \to \mathbb{R}^n \times \mathbb{R}^n$ are Lipschitz continuous functions and $\sigma\sigma^T$ is unif elliptic, $W := \{W_t \in \mathbb{R}^n, \mathcal{F}_t; 0 \le t < \infty\}$ $m\mathcal{F}_t$ BM
- $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the natural filtration of W_t
- crucially, an option is defined as the expected value of the payoff $h(\cdot)$ under $\mathcal{F}_t$

$$u(t,x) := \mathbf{E}_x[e^{-r(T-t)}h(\xi_T)|\mathcal{F}_t]$$

- for any function $u(t,x)$ a Cauchy final value problem for (1), with $h(x)$ being time T boundary data, is

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathcal{L}\right)u(t,x) &= 0, && \text{for all } (t,x) \in (0,T) \times D && (2) \\ u(T,x) &= h(x), && \text{for all } x \in D && (3) \end{aligned}$$

$$(\mathcal{L}f)(t,x) := \sum_{i=1}^{n} \mu^i(t,x)\frac{\partial}{\partial x^i}f(t,x) + \frac{1}{2}\sum_{i,k=1}^{n} a^{ik}(t,x)\frac{\partial^2}{\partial x^i \partial x^k}f(t,x) + c(t,x)f(t,x)$$

with

$$a^{ik}(t,x) := \sum_{j=1}^{n} \sigma^{ij}(t,x)\sigma^{kj}(t,x) = \Big(\sigma(t,x)\sigma^T(t,x)\Big)^{ik}.$$

Solving the Backward Kolmogorov equation for (2)-(3) gives:

$$u(t,x) = \mathbf{E}_x[e^{-r(T-t)}h(\xi_T)|\mathcal{F}_t] \tag{4}$$

Let $O_i =: u_i(t,x)$; (vega weighted) delta hedging the self-financing portfolio $Y_t := Y(t,\xi_t)$ gives

$$\begin{aligned} dY(t,\xi_t) &= w_0(t,\xi_t)dS_t + \sum_{i=1}^{m} w_i(t,\xi_t)dO_i \qquad Y_0 = y \in \mathbb{R} \\ &= \Big(\sum_{k=0}^{m} w_k(t,\xi_t)\nabla u_k\Big)\sigma dW_t \\ &= w(t,\xi_t)^T J(t,\xi_t)\sigma(t,\xi_t)dW_t, \qquad J := \begin{bmatrix} \nabla u_0 \\ \cdots \\ \nabla u_m \end{bmatrix}. \end{aligned} \tag{5}$$

Given $H_t \in m\mathcal{F}_T$, exercise value of an arbitrary contingent claim, Martingale Representation gives

$$\begin{aligned} H_t(\xi_t) &= \mathbf{E}[H_0] + \int_0^t w^T(s,\xi_s)J(s,\xi_s)\sigma(s,\xi_s)dW_s, \qquad \text{for all } t \geq 0 \\ &=: \mathbf{E}[H_0] + \int_0^t \mathfrak{W}^T(s,\xi_s)dW_s, \end{aligned}$$

implying $H_0 \in \mathbb{R}$. The map $\mathfrak{W}$ corresponds to a linear combination of contracts spanning the market filtration and the trading strategy map $w(\cdot,\cdot)$ is constructively given by

$$w^T(t,x) = \mathfrak{W}^T(t,x)\sigma^{-1}(t,x)J^{-1}(t,x), \qquad \text{for all } (t,x) \in [0,T] \times D.$$

APPROPRIATE FAMILY OF FACTOR PROCESSES

- References:
  - [Duffie and Kan, 1996]
  - [Duffie et al., 2003]
  - [Bru, 1991]
  - [Filipović and Mayerhofer, 2009]
  - [Cuchiero et al., 2010]
  - [Graczyk and Mayerhofer, 2012]
  - [da Fonseca et al., 2008]

- The idea of a factor model comes from the wonderful [Duffie and Kan, 1996].
- We are masters of the model, i.e. full freedom to trade empirical adequacy for process richness and analytical tractability.
- Requirements:
  - Affine form of the probability model of the state vector ξ_T;

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  - Factors cross correlation;
  - Filtration cardinality: minimal parameterization & adequate number of drivers;

▶ AFFINE FORM OF THE STATE VECTOR PROBABILITY MODEL

Definition

On $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{Q})$ and for a state space $D \subset \mathbb{R}^n$ let a generic stochastic process

$$\xi := \{\xi_t \in D, \mathcal{F}_t; 0 \leq t < \infty\}$$

and let $a, b \in \mathbb{R}^n$, $n \in \mathbb{N}$, $y \in \mathbb{R}$, $T \in (0, \infty)$ and $\langle\cdot,\cdot\rangle$ the usual inner product. Define:

$$G_{a,b}(y; x, t, T) := \mathbf{E}[e^{\langle a,\xi_T\rangle} \mathbb{1}_{\{\langle b,\xi_T\rangle \leq y\}} | \xi_t = x], \qquad \text{for all } x \in D. \tag{6}$$

Theorem ([Duffie et al., 2000])

A vanilla option payoff can be expressed as a linear combination of components of the type (6). With $a, b \in \mathbb{R}^n$, $v \in \mathbb{R}$, the Fourier transform with parameter v of any of these components, i.e.

$$\mathcal{G}_{a,b}(v; x, t, T) := \int_{\mathbb{R}} e^{ivy} G_{a,b}(dy; x, t, T), \qquad \text{for all } x \in D$$

is well defined. Choose now $n \in \mathbb{N} : n \geq 2$, $t \leq T : T \in (0, \infty)$. Define the map

$$\Psi : \mathbb{C}^n \times D \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C}$$

characterizing the family of integral transforms of ξ_T conditional on $\mathcal{F}_t$

$$\Psi(\gamma, x, t, T) := \mathbf{E}[e^{\langle\gamma,\xi_T\rangle} | \xi_t = x], \qquad \gamma \in \mathbb{C}^n$$

It holds

$$\mathcal{G}_{a,b}(v; x, 0, T) = \Psi(a + ivb, x, 0, T).$$

- FACTORS CROSS CORRELATION & DEGREES OF INTERACTION

- [Filipović and Mayerhofer, 2009, Thm. 2.2]: X^x affine iff $a(x)$, $b(x)$ affine in x
- Cross correlated CIR diffusions (#2) on canonical state space $\mathcal{X} := \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ will not do:

$$\begin{aligned} d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{bmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{x_2} \end{bmatrix} d\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \\ &= \begin{bmatrix} \sqrt{x_1} & 0 \\ \rho\sqrt{x_2} & \bar{\rho}\sqrt{x_2} \end{bmatrix} d\begin{pmatrix} W_1 \\ W_1^\perp \end{pmatrix} \\ &=: \Sigma(x_1, x_2) d\begin{pmatrix} W_1 \\ W_1^\perp \end{pmatrix} \end{aligned}$$

where $\langle W_1, W_2 \rangle = \rho t$, $\bar{\rho} := \sqrt{1-\rho^2}$ and

$$a(x_1, x_2) := \Sigma(x_1, x_2)\Sigma(x_1, x_2)^T = \begin{bmatrix} x_1 & \rho\sqrt{x_1 x_2} \\ \rho\sqrt{x_1 x_2} & x_2 \end{bmatrix}$$

is not an affine function - so the characteristic function will not have affine form.

- FILTRATION CARDINALITY & ADEQUATE NUMBER OF DRIVERS

- A characterization of the market in the model established so far faces the obvious problem of identification, i.e. of determining the minimal number of securities adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$.
- This number will be called *filtration multiplicity* or filtration cardinality and explicitly formalized in [Davis and Varaiya, 1974]
- An empirical question, best solved with PCA/ICA
- Lately, some purely probabilistic advances: [Jacod et al., 2008]

- Series of spot, RR, BFLY and ATM points for various currency pairs
- Emerging markets and more stable currencies; series examined stretching in strike up to 25 delta and in time from a month up to 5 years; observation span is 4 years
- Principal Components to unscaled data, i.e. Covariance matrices
- For each surface, the total variance explained by different components in the eigenspace has been analyzed in detail, testing for persistence in time of latent factors. While currencies pairs with marked skew (e.g. USDJPY) have performed distinctly better in the analysis than other pairs, the standard account of a level, a slope and a curvature effect has been found to be present across most of the pairs, with percentages of explained variances being reasonably stable in time.

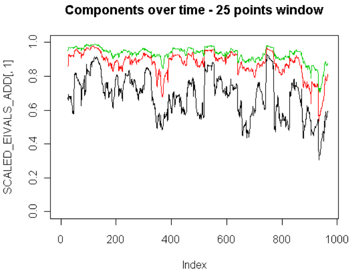

Figure: Prin. components on cumulative variation (%): USDJPY - 25 days windows

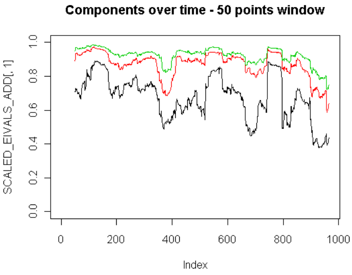

Figure: Prin. components on cumulative variation (%): USDJPY - 50 days windows

- Take $(B_t^1, \cdot, B_t^k)^T$ a k-dimensional BM and a initial value $y := (y_i)_{i=1}^k$: the process

$$\begin{aligned} X_t &:= (y + \sqrt{\Sigma} B_t)^T (y + \sqrt{\Sigma} B_t), \qquad X_0 = x = y^T y \geq 0 \qquad (7) \\ X_t &\sim \Gamma(k/2, x; 2\Sigma t) \end{aligned}$$

- Process (7) is denoted as "Square Bessel" ([Göing-Yaeschke and Yor, 2003])
- **Wishart process** (WP) is the generalization of (7) to matrix case
- State space $\overline{S}_d^+$ the cone of positive semidefinite matrices:

$$d\Sigma_t = (2pQ^T Q + M\Sigma_t + \Sigma_t M^T)dt + (\sqrt{\Sigma_t} dW_t Q + Q^T dW_t^T \sqrt{\Sigma_t}), \quad \Sigma_0 = \sigma_0 \in \overline{S}_d^+$$

with

$$Q, M \in M(d,d), \qquad p \geq 0$$

and

$$W := \{W_t := (W_t^{(1)}, ..., W_T^{(d^2)}), \mathcal{F}_t; 0 \leq t < \infty\}.$$

The Markov state vector ("factor process")

On $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{Q})$, define two stochastic processes

$$X := \{X_t \in \mathbb{R}, \mathcal{F}_t; t \in [0,\infty)\} \qquad \Sigma := \{\Sigma_t \in \overline{S}_d^+, \mathcal{F}_t; t \in [0,\infty)\}, \quad d \in \mathbb{N}$$

by the following pair of SDEs:

$$dX_t = \left(r - \frac{1}{2}\mathrm{Tr}[\Sigma_t]\right)dt + \mathrm{Tr}\left[\sqrt{\Sigma_t}dZ_t\right], \qquad X_0 = x_0 \in \mathbb{R} \quad (8)$$

$$d\Sigma_t = (2pQ^TQ + M\Sigma_t + \Sigma_t M^T)dt + (\sqrt{\Sigma_t}dW_tQ + Q^TdW_t^T\sqrt{\Sigma_t}), \quad (9)$$
$$\Sigma_0 \in \overline{S}_d^+$$

$W := \{W_t := (W_t^{(1)}, \dots, W_t^{(d^2)}), \mathcal{F}_{t\in[0,\infty)}\}$ and $Z := \{Z_t := (Z_t^{(1)}, \dots, Z_t^{(d^2)}), \mathcal{F}_{t\in[0,\infty)}\}$ are $m\mathcal{F}_t$-matrix valued BM, $Q, M \in M(d,d)$, $p \geq \frac{d+1}{2}$, $r \in \mathbb{R}$.

Define, for some $R \in M(d,d)$,

$$Z_t := W_tR^T + B_t\sqrt{I_d - RR^T}, \qquad B := \{B_t := (B_t^{(1)}, \dots, B_t^{(d^2)}), \mathcal{F}_t; 0 \leq t < \infty\}$$

where d^2 components of W and d^2 components of B are independent. Rewrite (8)-(9) as

$$X_t = X_0 + \int_0^t \left(r - \frac{1}{2}\mathrm{Tr}[\Sigma_s]\right)ds + \int_0^t \mathrm{Tr}\left[\sqrt{\Sigma_s}\left(dW_sR^T + dB_s\sqrt{I_d - RR^T}\right)\right] \quad (10)$$

- Requirements are met:
  - Affine form of the probability model of the state vector (here: X_T):

$$\begin{aligned} \Psi(\gamma, x, t, T) &:= \mathbf{E}\left[e^{\langle \gamma, X_T \rangle} \middle| X_t = x \right], \qquad x \in \mathbb{R}, \gamma \in \mathbb{R} \\ &= \exp\left(\mathrm{Tr}[A(T-t)\Sigma_t] + b(T-t)X_t + c(T-t) \right)\Big|_{X_t = x} \end{aligned} \tag{11}$$

$A : \mathbb{R}^d \mapsto \mathbb{R}^d$, $b : \mathbb{R}^d \mapsto \mathbb{R}$, $c \in \mathbb{R}$ are obtained by solving Riccati ODEs ;

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  - Factors cross correlation: see [da Fonseca et al., 2008]

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  - Factors cross correlation: see [da Fonseca et al., 2008]
  - Filtration cardinality & minimal parameterization:

Proposition (Minimal parameterization [Trabalzini, 2015, Prop. 3.27])

Let $d \in \mathbb{N}$ s.t. the volatility process $\Sigma_t \in \overline{S}_d^+$, for all $t > 0$ in (9), i.e.

$$d\Sigma_t = (2pQ^TQ + M\Sigma_t + \Sigma_t M^T)dt + (\sqrt{\Sigma_t}dW_tQ + Q^TdW_t^T\sqrt{\Sigma_t}), \qquad \Sigma_0 = \sigma_0 \in \overline{S}_d^+.$$

It has $2d^2$ independent Brownian drivers. Given symmetry, only $\frac{d(d+1)}{2}$ are indeed necessary. Moreover, let K be the probability filtration multiplicity, as defined above where $K = 1 + \frac{d(d+1)}{2}$. Then the cardinality of (10) can be reduced to one plus the cardinality of (9), thus matching K.

Theorem (Uniqueness of PDE classical solution [Trabalzini, 2015, Thm. 3.38])

Let D be a open connected subset of $\overline{S}^{+}_{d\times d+1}$ and $T \in (0,\infty)$. For given measurable functions

$$h : D \mapsto [0,\infty), \quad g : [0,T]\times D \mapsto (-\infty, 0] \quad c : [0,T]\times D \mapsto \mathbb{R}$$

take $u : [0,T]\times D \mapsto [0,\infty]$ as identified by the stochastic representation (4) and consider the initial value problem (2)-(3), where the process entering the stochastic representation is given by

$$dX_t = \left(r - \frac{1}{2}\mathrm{Tr}[\Sigma_t]\right)dt + \mathrm{Tr}\left[\sqrt{\Sigma_t}\left(dW_t R^T + dB_t\sqrt{I_d - RR^T}\right)\right], \quad X_0 = x_0 \tag{12}$$

and the infinitesimal generator $\mathcal{L}_{X,\Sigma}$ is given by

$$\mathcal{L}_{X,\Sigma} := \left(r - \frac{1}{2}\mathrm{Tr}[\Sigma]\right)\frac{\partial}{\partial x} + \frac{1}{2}\mathrm{Tr}[\Sigma]\frac{\partial^2}{\partial x^2} + \mathrm{Tr}[(2pQ^T Q + M\Sigma + \Sigma M^T)\underline{D} + 2\Sigma\underline{D}Q^T Q\underline{D}] + 2\mathrm{Tr}[\Sigma RQ\underline{D}]\frac{\partial}{\partial x}$$

If $n := 2p \geq d+1$ and $(2pQ^T Q + M\Sigma_t + \Sigma_t M^T)$ in (9) is positive semidefinite, $u(t,x)$ satisfies the IVP above, $u(t,x) \in C^{1,2}([0,T]\times D)$ and the classical solution to the PDE is unique.

ENCODING COMPLETION & DYNAMIC HEDGING

- Main references:
  - [Davis and Obłój, 2008]
  - [Kramkov and Predoiu, 2014]

- Necessary condition for the invertibility of the market matrix is

Assumption (ii) - Full rank market Jacobian

For all $(t, x) \in [0, T] \times D$, $J(t, x)$ in eq. (5) has full rank

- Assumption quite difficult to verify. By imposing additional structure on the maps that appear in the final value problem in (2)-(3) a sufficient condition holds:

Theorem ([Davis and Obłój, 2008, Thm. 4.2])

Verification of Ass. (ii) over the full time-space domain can be reduced to its verification in a single point $(t_0, x_0) \in [0, T] \times D$ under:

1. *(Conditions on factor process): Eq. (1) has unique strong solution, $\sigma(t, x)\sigma(t, x)^T > \epsilon I$, for some ϵ, for all $(t, x) \in (0, T) \times D$ (strong ellipticity) and μ, σ are such that $\xi := \{\xi_t \in D, \mathcal{F}_t; 0 \leq t \leq T\}$ admits a positive density on D, $t \leq T$;*
2. *(Condition on payoff maps): $h_i : D \mapsto [0, \infty)$ have at most polynomial growth, $1 \leq i \leq n$;*
3. *(Conditions on PDE classical solutions): $u_i : [0, T] \times D \mapsto [0, \infty]$, $1 \leq i \leq n$ are the unique solutions of eqs. (2)-(3) with at most polynomial growth; assume moreover that μ, σ, h_i are such that u_i, are real analytic functions.*

- The target is to explicitly compute the trading strategy map

$$w^T(t,x) = \mathfrak{W}^T(t,x)\sigma^{-1}(t,x)J^{-1}(t,x), \qquad \text{for all } (t,x) \in [0,T] \times D \quad (13)$$

as a linear combination of existing contracts spanning the market filtration
- By Theorem (4), condition on Jacobian needs being verified at a single point

Theorem (Market completion - full form [Trabalzini, 2015, Prop. 4.8])

Assumption (ii) is verified for the model whose factor process is given by eqs. (8)-(9) *hence a market completion exists.*

- The theorem is proved via proposition below, which needs some auxiliary results

Proposition (Market completion - simpler case [Trabalzini, 2015, Thm. 4.9])

For a factor process of the form in equations (12)*, i.e.*

$$dX_t = \left(r - \frac{1}{2}\mathrm{Tr}[\Sigma_t]\right)dt + \mathrm{Tr}\left[\sqrt{\Sigma_t}(dW_t R^T + dB_t\sqrt{I_d - RR^T})\right], \quad X_0 = x_0 \in \mathbb{R}$$

where matrices are diagonal, the trading strategy in (13) *can be explicitly computed and hence a market completion exists. It can be encoded in the structure of the coefficients of the corresponding matrix Riccati equations solving the Wishart stochastic variance process* (12)*.*

Sketch of proof: analytic representation of the market matrix & approximation argument

Consider again the Market Matrix $J : [0, T] \times D \mapsto \mathbb{R}^{TM+1}$ in (5) where setting $m := TM + 1$ and exploding vector $x := (x^1, \dots, x^n)^T$ gives

$$J_{(m,n)}(t,x) = \begin{bmatrix} \nabla u_{T,0}(t,x) \\ \cdots \\ \nabla u_{1,m-1}(t,x) \end{bmatrix} := \begin{bmatrix} \nabla u_{T,0}(t,(x^1,\dots,x^n)^T) \\ \cdots \\ \nabla u_{1,m-1}(t,(x^1,\dots,x^n)^T) \end{bmatrix}, \quad \text{for all } (t,x) \in [0,T] \times D.$$

Proposition (Reduction to analytic components [Trabalzini, 2015, Prop. 4.10])

Let $a_j \in \mathbb{R}^n$, $n \in \mathbb{N}$ and $\forall j \in \{0, \dots, m-1\}$ assume that the market trades products of the form

$$\begin{aligned} u_{t+\tau,j}(t,x) &= u_{t+\tau,j}(t,(x^1,\dots,x^n)^T) \\ &:= \mathbf{E}[e^{\langle a_j, \xi_{t+\tau}\rangle} | \xi_t = x], \qquad \text{for all } x := (x^1,\dots,x^n)^T \in D \end{aligned}$$

where $\xi := \{\xi_t := (\xi_t^1, \dots, \xi_t^n)^T \in D, \mathcal{F}_t; 0 \le t < T\}$ is a Wishart-type factor process with no spot component. Then the market matrix $J_{(m,n)}$ can be expressed in terms of conditional MGF and has components that are analytic.

Proposition (Spanning Laplace Transforms [Trabalzini, 2015, Prop. 4.11])

For any payoff $H : D \mapsto \mathbb{R}_+$, $H \in L_2(D)$ there exists a sequence $\left(\gamma_n, t_n, T_n, w_n\right)_{n\in\mathbb{N}}$ with $w_n \in \mathbb{C}$:

$$H(x) = \sum_{n \ge 0} w_n \Psi(\gamma_n, x, t_n, T_n), \qquad \text{for all } x \in D$$

where $\Psi(\gamma, x, t, T)$ is the Laplace Transform in (11).

RECURSIVE CALIBRATION & THE NON-LINEAR FILTERING

- Main references:
  - [Picard, 1991]
  - [Schwartz, 1969]
  - [Bell and Cathey, 1993],

- Once performed, the calibration needs not to be repeated in time, from t to $t+\Delta t$;
- Observing a vector of option prices at each time t_i $i \in \{1, \dots, \}$ gives access to a surface of option prices across the strike and maturity ladder;
- Modelling the deterministic set of market prices at time t by the ODE

$$dY_t = h(y;\xi_t, t, T)dt, \tag{14}$$

where $h(\cdot)$ is the observation function, the connection with non-linear filtering can be performed via introducing some noise in the observation process, as

$$dY_t = h(y;\xi_t, t, T)dt + \epsilon dB_t, \qquad Y_0 = y_0 \in \mathbb{R}^n, \quad \epsilon > 0.$$

- The small noise filtering comes from a *bona fide* small noise, in the sense that what is really observed is an ODE, but phenomena like quantization, transaction cost, bid-ask spread lead to a stochastic treatment;
- The computational advantage of a well-understood computational strategy as EKF will be revealed in the following, where the two stages above (i) prediction and (ii) correction will be interpreted as (i) convolution & basis function expansion and (ii) Gauss-Newton optimization for inversion of the Jacobian.

The filtering problem consists in the two sequential steps:

1. Prediction (Extrapolation) step : evolving Kolmogorov Forward Equation;
2. Correction (Update) step: applying conditional Bayes Formula.

Theorem (Prediction by convolution & convergence [Trabalzini, 2015, Prop. 6.2])

Define by $f(\cdot)$ the Chapman-Kolmogorov equation for the prediction step. It can then be interpreted as the convolution

$$f(x) := \int_{-\infty}^{\infty} d\alpha(z) g(x - z/\sigma) \tag{15}$$

where $\alpha(\cdot)$ is the distribution of the factor process ξ_t and $g(x; \mu, \sigma)$ a Gaussian density. Moreover, $f(\cdot)$ admits the asymptotically converging estimate

$$\hat{f}_n(x) := \frac{1}{n} \sum_{i=1}^{n} \hat{a}_{in} \varphi_i(x), \qquad \hat{a}_{in} := \frac{1}{n} \sum_{i=1}^{n} \varphi_i(x_j),$$

where $\varphi(\cdot)$ are Hermite basis functions.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $x_k \in \mathbb{R}^n$, $n \in \mathbb{N}$ and let $x_\star$ be the accumulation point. Define seq. $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = x_k - [\nabla f(x_k)]^{-1} f(x_k). \tag{16}$$

The sequence $\{x_k\}_{k=0}^{\infty}$ has quadratic convergence, i.e.

$$||x_{k+1} - x_\star|| \leq C_0 \, ||x_k - x_\star||^2 , \qquad C_0 \in (0, \infty).$$

Take now $f := \nabla J$ where $\nabla J : \mathbb{R}^n \mapsto \mathbb{R}^n$. Eq. (16) becomes

$$x_{k+1} \quad = \quad x_k - [\Delta J(x_k)]^{-1} \nabla J(x_k) \tag{17}$$

For $x \in \mathbb{R}^n$, the optimization above can be connected to the least squares minimization problem of the target function f, i.e.

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2} \, ||r(x)||^2 , \qquad ||x|| := \sqrt{x^T x}$$

whose Jacobian and Hessian are, respectively

$$\nabla f(x) \quad = \quad r^{'}(x)^T r(x) \tag{18}$$

$$\Delta f(x) \quad = \quad r^{'}(x)^T r^{'}(x) + r^{''}(x)^T r(x). \tag{19}$$

Using (18) and the approximation $\Delta f(x) = r^{'}(x)^T r^{'}(x)$ in (19), eq. (17) becomes

$$x_{k+1} = x_k - [r^{'}(x_k)^T r^{'}(x_k)]^{-1} r^{'}(x_k)^T r(x_k). \tag{20}$$

Proposition (Recursive calibration: Non-linear filter formulation [Trabalzini, 2015, Prop. 6.01])

On $(\Omega, \mathcal{F}, \mathbb{Q})$, consider the two stochastic processes (resp. the state and the observation process)

$$\xi := \{\xi_t \in D, \mathcal{F}_t; 0 \le t < \infty\} \qquad Y := \{Y_t \in \mathbb{R}^n, \mathcal{F}_t; 0 \le t < \infty\}$$

given by the pair of SDEs

$$d\xi_t = \mu(t, \xi_t)dt + \sigma(t, \xi_t)dW_t, \qquad \xi_0 = x \in D, \tag{21}$$

$$dY_t = h(y; \xi_t, t, T)dt + \epsilon dB_t, \qquad Y_0 = y_0 \in \mathbb{R}^n, \quad \epsilon > 0 \tag{22}$$

where $\mathbb{R}^n \ni y := \{y_k\}^T$ is the vector of strikes, $W := \{W_t \in \mathbb{R}^n; \mathcal{F}_t; 0 \le t < \infty\}$ $B := \{B_t \in \mathbb{R}^n; \mathcal{F}_t; 0 \le t < \infty\}$ are $\mathbb{R}^n$ BMs, $\{\mathcal{F}_t\}_{t \in [0,T]}$ is their natural filtration, $B \perp W$ and

$$h : \mathbb{R}^n \times D \times [0, T] \times [0, \infty) \mapsto \mathbb{R}^n$$

is given by

$$\begin{aligned} h(y; x, t, T) &= \{h_\alpha(y_k; x, t, T)\}^T, \qquad k \in \{1, \dots, n\} \\ &:= \left\{ \mathbf{E}\left[\left(y_k e^{\langle 0, \xi_T \rangle} - e^{\langle a, \xi_T \rangle} \right) \mathbb{1}_{\{\langle a, \xi_T \rangle \le \ln y_k\}} \middle| \xi_t = x \right] \right\} \end{aligned}$$

where $a \in \mathbb{R}^n$, $T \in (0, \infty)$ and $\langle \cdot, \cdot \rangle$ the usual inner product.
Then the calibration problem can be solved via the non-linear filter (21)-(22) based on (i) the convolution in eq. (15) and (ii) the Gauss-Newton recursion in (20).

- **Conclusions**
  In this talk, market completion - i.e. using information contained in a spanning set of traded assets for pricing and hedging purposes, has been investigated in an entirely novel way. A rich but tractable class of factor processes based on the Wishart family has been singled out and computable conditions have been furnished under which any contingent claim in the market can be replicated by a combination of options and the forward. The dynamic hedging problem under scrutiny has further been connected in a novel way to non-linear filtering techniques and Gauss-Newton optimization ideas.
- **Addenda**
  - Numerics for stability of dynamic hedging strategies
  - Application in automarket: the opaque nature of Wishart based processes parameters is no issue for automatic markets (FX space)
  - Applications in Risk Management
- **Open questions**:
  - Connect the thread on filtration cardinality with the work on Estimation of Brownian dimension of a continuous Itô process ([Jacod et al., 2008])
  - ELMM vs EMM for matrix valued processes: towards a general theory

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