On Certain Tendencies in the Development of Mathematics*

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Translated from the Russian by A. Shenitzer

Every being tends to take its habitat for granted, to regard it as something which cannot be different and therefore gives rise to no questions. The attitude of the mathematician to his discipline is no different. It is only on those rare occasions when the mathematician looks at mathematics from the outside that he notices immediately what a strange and implausible phenomenon he has been dealing with all his life. For me such an occasion was the flattering invitation to say something about mathematics to colleagues engaged in areas of learning far removed from it.

Viewed superficially, mathematics is the result of centuries of effort by many thousands of largely unconnected individuals scattered across continents, centuries and millennia. However, the internal logic of its development much more resembles the work of a single intellect developing its thought in a continuous and systematic way, and only using as a means a multiplicity of human individualities, much as in an orchestra playing a symphony written by some composer the theme moves from one instrument to another so that as soon as one performer is forced to cut short his part, it is taken up by another player, who continues it with due attention to the score.

Truly, this is not a figure of speech! The history of mathematics is full of examples in which the discovery of one scholar remained unknown only to be reproduced later with amazing accuracy by another. In a letter written the night before the duel which led to his death, Galois formulated some fundamental assertions about the integrals of algebraic functions. More than twenty years later, Riemann, who certainly did not know of Galois' letter, posed a new and proved those very same assertions. To give another example: After Lobachevsky and Bolyai independently founded noneuclidean geometry, it came to light that Gauss and Schweikart had independently arrived at the same results more than ten years earlier. One experience a strange feeling when one sees the same diagrams, drawn as if by the same hand, in the writings of four mathematicians working independently of one another.

One is drawn to the conclusion that such striking and mystifying human activity, spanning several millenia, far from being an accident, must have a purpose. Granted this premise we must pose the question: What is this purpose?

How can a whole discipline, not just one of its chapters and not just one phase of its development, have a single purpose? The example of physics, always so closely connected with mathematics, may suggest an answer. By Newton's time physicists conceived of a glorious aim. The aim was to construct a theory (or, in the parlance of the time, a system) of the world that was to embrace the universe in a few simple laws from which it would be possible to deduce the multiplicity of physical phenomena by purely logical means. For a long time it seemed that Newton solved this problem in principle and that it had fallen to his followers merely to check that his system accounted for all known phenomena. The sole exception lay on the periphery of physics: the theory of electricity did not fit the newtonian scheme. In the 19th century however, it was precisely electromagnetic phenomena which came to occupy a central position in physics; and while this fact shook the newtonian conception, it also gave rise to the hope that newtonian mechanics, supplemented by Maxwell's theory of the electromagnetic field, would yield a complete and final picture of the universe. But these expectations, too, were to end in disappointment. Soon quantum mechanics and the theory of relativity broke up all old conceptions. For a time physicists hoped to deduce from field theory alone or from relativistic quantum mechanics a complete theory of elementary particles and a a new system of the universe. So far this has not come to pass and it is doubtful that many physicists regard such hopes as realistic. At any rate, should physicists at any time again achieve a degree of unity as to the physical picture of the universe, it will be difficult, given the many previous revisions, to believe in the finality of the resulting system.

Returning to mathematics, we must admit that it has never formulated a global aim of the kind which ambitious physics has formulated, but not realized, on a number of occasions. How is this fact reflected in its development?

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Mathematics grows impetuously and continuously, undisturbed by the rebuilding and crises typical of physics, constantly enriching us with new ideas and concrete facts. I am fully convinced that the achievements of modern mathematics are no less perfect than the classics of the 19th, 18th and 17th centuries, and that they can even bear comparison with the fruits of Hellenic genius. However, even the finest of modern accomplishments are not superior, in principle, to the classical achievements! What is the value of of the unlimited accumulation of ideas which are, in principle, equally profound? Does not mathematics become a strikingly beautiful variant of Hegel's "idiotic infinity"?

Any activity devoid of purpose is by this very fact devoid of sense. And if we equate humanity with a living organism, then mathematics does not resemble an intelligent activity directed towards an aim; but rather it becomes analogous to instinctive actions, which repeat one another, like copies, as long as some external or internal stimulus is at work.

Without purpose, mathematics cannot even work out an idea of its form. The ideal it is left with is uncontrolled growth, or more precisely, extension in all directions. To use another comparison we might say that the development of mathematics resembles not the growth of a living organism which retains its form and alone determines its boundaries but rather the growth of a crystal or the diffusion of a gas. Both crystal and gas will spread without limit unless checked by an external obstacle.

Clearly, such a development of mathematics contradicts the sense of intelligence and beauty inevitably gained from contact with the subject, just as there is an inherent contradiction in the concept of a symphony which goes on for ever.

Is ours the only discipline which gives rise to this problem? I do not think that mathematics differs radically from other forms of cultural activity. True, the entities mathematics deals with are more abstract; it rejects more accidental properties. As Plato said, there is in mathematics more knowledge of pure being and fewer opinions about objects in the visible world; in mathematics "one seems to dream of essence." That is why tendencies clearly discernible in mathematics, while universal, are dimly discernible in other fields of knowledge. In particular, the absence of aims and form of which we spoke above is found, I think, in practically all aspects of the life of modern man. Thus, the aimless development of mathematics has been paralleled, as we have seen, by the way in which physics, pursuing a seemingly unattainable goal, lost the idea of any goal whatever.

For several centuries now man has been in the grip of feverish activity, formless and devoid of all aim and meaning save that of unlimited growth. This was called "progress" and for a time it became something of a substitute for religion. Its latest offspring is modern industrial society. It has been frequently pointed out that this race is self-contradictory and leads to catastrophic material consequences: to an ever-increasing tempo of life straining human capacities, to overpopulation and to the destruction of the environment. Using mathematics as an example I want to call attention to spiritual consequences which are no less destructive: all human activity loses a global aim and becomes meaningless.

The danger here is more than just negative in nature. It does not simply consist in the fact that the intense efforts of humanity, the life of its most talented members, are ultimately devoid of meaning. The full danger lies not only in our inability to predict the consequences of actions whose purpose we do not understand. Such is the spiritual constitution of mankind that it cannot for long reconcile itself to an activity whose aim and sense elude it. Here as well as in many other phenomena, what begins to operate is the mechanism of substitution: when unable to find what they need, human beings resort to a substitute. We are all well-acquainted with a relevant example. Having broken away from the God of mercy and love, men have at once fashioned other gods which demand millions of human victims. In accordance with this law, human cultural activity, if deprived of a clear understanding of its aims, attempts to borrow meaning from other sources. In particular, a mathematician may seek the purpose of his work in filling the order of a state, for which he is ready to compute the trajectory of a rocket, design an eavesdropping apparatus, or, if he is exceptionally capable, plan a whole society of hybrids – part-man, part-computer. It is not just the souls of scientists that are mutilated by such an order of things. There appear whole areas of mathematics devoid of that divine beauty which captivates all those familiar with our discipline.

More than two thousand years of history convince us that mathematics is unable to formulate the aim necessary to direct its own development. It must therefore borrow that aim from without. Obviously, I do not intend to try to solve this profound problem which involves not only mathematics but all human endeavours. I merely wish to indicate basic directions where one might search for a solution.

There are, it seems, two such directions. One could try to derive the aim of mathematics from its practical applications. However, it is hard to believe that the justification of a higher, spiritual activity could be found in a lower, material activity. In a copy of the "Gospel of Thomas" discovered in 1945 Jesus says with irony:

> "If the flesh was made for the spirit, it is a miracle. But if the spirit was made for the body, it is a miracle of miracles."

The history of mathematics proves convincingly that there is no "miracle of miracles." If we consider the decisive moment in the history of mathematics, when it took its first and most significant step for humanity, when logical proof, the very basis of mathematics, came into being, then we see that the subject matter involved simply ruled out any practical applications. The first theorems of Thales established truths obvious to every sensible person, such as the diameter of a circle dividing it into two equal parts. It took genius not to prove such results but to understand that these results required proof. The practical value of such discoveries is obviously nil.

Notwithstanding the variety and depth of the applications of mathematics at the present time, it is again definitely not the case that these applications inspired its most beautiful achievements. This being so, one could hardly expect applied mathematics to furnish the purpose which mathematics was not itself able to find.

If, then, we reject this path, I think there remains just one possibility. The purpose of mathematics cannot be derived from an activity inferior to it but from a higher sphere of human activity, namely, religion.

Clearly, it is very difficult at the present moment to see how this can happen. But it is even more difficult to conceive how mathematics can go on developing forever without knowing what it studies and why. It is bound to perish in the next generation, drowned in a flood of publications – but that is, after all, only the most elementary external reason.

On the other hand, the suggested solution, as history itself has proved, is in principle possible. If we again go back to the time when mathematics came into being, we see that in then knew its purpose and that it acquired that purpose by following precisely this path. Mathematics as a science came into being in the 6th century B.C. in the religious community of the Pythagoreans and was part of their religion. Its aim was clear. By revealing the harmony of the world as expressed in the harmony of numbers it provided a path leading towards a union with the divine. It was this lofty aim which at that time supplied the forces necessary for a scientific feat to which in principle there can be no equal. What was involved was not the discovery of a beautiful theorem, not the creation of a new branch of mathematics, but the creation of mathematics itself.

Then, almost at the moment of its birth, those properties of mathematics had already come to light thanks to which general human tendencies are more clearly apparent therein than anywhere else. This is precisely the reason why at that time mathematics served as a model for the development of the fundamental principles of deductive science.

In conclusion I wish to express the hope that for this same reason mathematics can now serve as a model for the solution of the fundamental problem of our time:

To reveal a supreme religious aim and purpose for mankind's cultural activity.

Translator's note. I first prepared this translation in 1975. It was then read by G. Brooke who suggested a number of improvements. Since that time I have shown the translation to many friends, and some of them, namely M. Jenkins, H. Grant, and V. Cohn, have suggested additional improvements.

Abe Shenitzer